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Dispersion in Momentum Space and the Existence of $U(t,t')$

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Abstract

The operator $U(t,t')$ giving transition probabilities between finite times or connecting free and interacting fields does not exist (apart from the ultraviolet divergence problem) because of the 3-translation invariance of current quantum field theory. To remedy this, the idealization that one has an infinite time $T = \infty$ to prepare initial, or measure final, n-particle momentum eigenstates is discarded here. It is shown that random space-time (which itself eliminates ultraviolet divergences from field theory) implies and fixes uniquely a random momentum space if free particle momenta K are determined by time-of-flight measurements with $T < \infty$. In particular, the dispersion of $K \propto m\lambda/T$, where λ is the space-time dispersion and m is the particle mass. Stochastic momentum space is incorporated into field theory in a preliminary way; because 3-translation form-invariance is slightly violated, the unitary U-operator expressed as the usual T-exponential exists and the limit $U \to S$ as $t \to \infty$, $t' \to -\infty$ is well-defined without *ad hoc* tricks like the adiabatic cut-off. A frame-dependence is necessarily introduced into fields and U-operator, and the transformation properties expressing Lorentz covariance are of the same more general type encountered in previous work on quantum field theory over stochastic space-time.

1. *Introduction*

It is well known that the U operator which connects the interacting Heisenberg Picture field and the free ('in') field (Bjorken & Drell,1965; Schweber, 1961a), or gives the transition probability between states at finite times (Jauch & Rohrlich, 1955; Schweber, 1961b), does not exist in the current framework of QFT (quantum field theory). Apart from the ultraviolet divergence difficulty, there is another difficulty connected with 3-translation invariance which manifests itself in perturbation theory by the appearance of three-dimensional delta functions $\delta(\mathbf{0})$ in some matrix elements. A sketch of the formal proof that U does not exist goes like this. From the usual axioms one proves (Haag, 1962) that there can be only one *homogeneous* state $\Omega_0 \in$ state vector Hilbert Space:

$$\mathbf{P}\Omega_0 = 0 \tag{1.1}$$

where $\mathbf{P}$ generates spatial translations in some frame. Now in QFT the vacuum Ω_0 is certainly homogeneous. Consider the state $\Omega \equiv U(t,t')\Omega_0$; since U is unitary and translationally invariant, we have $\mathbf{P}\Omega = 0$ and $\|\Omega\| = \|\Omega_0\| = 1$, so Ω is also homogeneous. But Ω is not the same state as Ω_0 if either t or t' is finite because, e.g. in Lagrangian field theory owing to the local structure of the interaction, U contains terms which create a set of particles from Ω_0. Thus Ω, and hence U, cannot exist, Q.E.D. Below, by an explicit calculation of the vacuum $\rightarrow$ one pair, one photon component of U in quantum electrodynamics, we shall show explicitly how this $\delta(\mathbf{0})$ emerges.

To some, in fact most, theoreticians today, this failure indicates that field theory must be thrown out *in toto*. The author has never been able to understand this drastic attitude. When one considers how naturally the U operator theory in T-exponential form *formally* solves the interacting QFT problem—it suffices here just to read through Sections 4-2 and 4-3 of Jauch and Rohrlich's book, or Section 17a of Schweber's book—it would seem much more promising just to look for the idealizations in present-day field theory which are making the mathematical formalism singular (the divergences, these $\delta(\mathbf{0})$, etc.) and remove them. That is the guiding idea of this article, which seeks a *physical* cure of this trouble by relaxing some of our rigid present idealizations. To look for a pseudo solution by expending much mathematical ingeniosity, necessitating close attention to mathematical rigor, while remaining in the present framework of axioms is the farthest thing from my intention.

And in fact, in a nutshell, here is the cure I propose. As one sees from the formal proof that U does not exist, and with even more insight from an actual calculation of a matrix element from Feynman graphs, the villain is 3-translation invariance. This is (slightly) destroyed—enough to make the formalism mathematically meaningful —by putting dispersion into momentum space, i.e., by making $P_\mu \equiv$ *4-momentum of a free particle* a random variable. Then it turns out, as we show below, that U exists and goes smoothly into S as $t \rightarrow +\infty$, $t' \rightarrow -\infty$ in virtue of the Riemann-Lebesgue Lemma, without the necessity of *ad hoc* tricks like the adiabatically switched off potential.†

But why should there be an intrinsic dispersion in momentum space? We assert that this follows from incorporating into the theory the two facts: (1) particle momentum is determined via the time-of-flight method through particle *position* measurements, and (2) we do

† Compare, say, Schweber (1961a), p. 322.

not dispose of an infinite time to make these measurements (which is the present idealisation) but in fact of a finite time T.

But now there is no limit to the sharpness with which we can measure P_μ by the time-of-flight method if the space and time co-ordinates of events can be measured sharply, even though we only have $T < \infty$ in which to work. On the other hand, even if space-time coordinate measurements show an intrinsic uncertainty $\sim\lambda$, P_μ can be measured with arbitrary sharpness by taking T long enough. (These are evident on a few moments' reflection; see also Section 4.) Thus for the standard deviation a of the random variable P_μ we must have $a \propto \lambda/T$ so that either for $\lambda \to 0$ or $T \to \infty$ we fall back on the present 'U-difficulty.'

Then one achieves a beautiful economy. Namely, the stochastic space-time† necessary to remove the ultraviolet divergences from U and its limit S determines uniquely the stochastic momentum space necessary to remove the U-difficulty (sometimes called also the 'Haag Theorem' difficulty), as we shall show below. The introduction of a stochastic momentum space is not an extra principle, we do not need to make extra assumptions, concerning its existence and form, independent of those of stochastic space-time provided only we take $T < \infty$.

Just as in the stochastic QFT theory, the cure proposed here avoids rather easily most of the troubles which plague nonlocal theories. One pays for this ease by a broadening of the invariance scheme which today is called 'Lorentz invariance'. As we have argued elsewhere in many places,‡ one keeps true, essential 'Lorentz invariance', namely the Relativity Principle, which states that the theory should prefer no Lorentz frame. One gives up something which may or may not be essential, but in any case is extraneous to the Relativity Principle. The fact is that a frame-dependence of fields, the S-operator, etc., is introduced, which has certain observational consequences which may be tested in certain very high-energy experiments.§ To date these crucial experiments have not been done.

† Ingraham, R. L. *Nuovo cimento*, **24**, 1117 (1962); **27**, 303 (1963). The second paper contains also corrections and emendations to the first. See also short expositions of the theory in applications to very high energy scattering, e.g. Bailey, D. and Ingraham, R. L. (1966). *Physical Review*, **152**, 1290, and in a forthcoming book, *Renormalization Theory of Quantum Field Theory with a Cut-off*, to be published by Gordon and Breach.

‡ E.g., Ingraham, R. L. (1962). *Nuovo cimento*, **26**, 328, especially Section 3, and the last two references of the preceding footnote.

§ Ingraham, R. L. (1967). ICTP Internal Report No. 11; also *Nuovo cimento*, **32**, 323 (1964); **39**, 361 (1965); and *Physical Review*, **152**, 1290 (1966).

Since we are groping for new physical ideas, not polishing an established theory to final mathematical perfection, our arguments will be presented in a sort of a relaxed physical exposition, with not much attempt at rigor or at answering all the questions which could be raised. Briefly then, the program of this paper is as follows. First we illustrate the U-difficulty on Feynman graphs, then propose *ad hoc* a modification $\delta \to \delta_T$ of the delta functions in the momentum space commutation relations. Necessary conditions on δ_T which make U physical and the limit $U \to S$ well defined are elucidated; these will be shown later to be demanded by Poincaré group symmetry and the positive definite metric of state vector space $\mathscr{H}$. Next, stochastic momentum space applied to QFT is developed *ab initio*, and the transformation properties of the stochastic mean fields $a(k;\mathscr{L})$ are found. δ_T is then derived from the more basic frequency function $f(k'-k)$ and shown to have the properties postulated above. After that, we assume that our random P_μ is determined in terms of a random X^μ by the time-of-flight method, and derive $f(k'-k)$ from stochastic space-time; it has all the properties demanded by the physical arguments sketched earlier. Next the bases in $\mathscr{H}$ of 'relatively free' particle states generated by the algebra of the $a^*(k;\mathscr{L})$ on the vacuum are looked at more closely; in particular they are no longer strictly orthonormal. Finally some preliminary thinking is given on the question of the relative size of various time parameters for optimum accuracy, and whether various limits $T \to \infty$, etc. make sense.

2. *The U-Difficulty*

Formal application of Lagrangian field theory yields the connection $\phi_{\text{in}}(x) = U(t,-\infty)\boldsymbol{\phi}(x)\,U(t,-\infty)^{-1}$ between the free, in-field and interacting Heisenberg Picture field at time $t \equiv x^4$. U has the T-exponential form

$$U(t,t') = T\exp\left(-i\int_{t'}^{t} \mathscr{H}_I(x)\,d^4x\right) \tag{2.1}$$

where $\mathscr{H}_I(x)$ is the interaction Hamiltonian density formed from the in-fields (Bjorken & Drell, 1965; Schweber, 1961). Consider the state $\Omega \equiv U(t,-\infty)|0\rangle$ where $|0\rangle$ is the vacuum; it is homogeneous, as explained in the Introduction. We calculate the $0(e)$ part $\omega^{(1)}$ of the

1-pair, 1-photon component ω of Ω in quantum electrodynamics, namely

$$\omega^{(1)} \equiv \int d^3p\, d^3q\, d^3k |\mathbf{p},\mathbf{q},\mathbf{k}\rangle\langle \mathbf{p},\mathbf{q},\mathbf{k}|\, U^{(1)}(t,-\infty)|0\rangle \tag{2.2}$$

The Feynman diagram gives immediately for the matrix element, with an obvious abbreviation

$$\begin{gathered}\langle 3|U|0\rangle = \delta(\mathbf{p}+\mathbf{q}+\mathbf{k})\frac{\exp(iEt-\varepsilon|t|)}{E}G(\mathbf{p},\mathbf{q}) \\ E \equiv E(\mathbf{p},\mathbf{q}) \equiv (\mathbf{p}^2+m^2)^{1/2}+(\mathbf{q}^2+m^2)^{1/2}+|\mathbf{p}+\mathbf{q}|, \qquad \varepsilon \to 0+\end{gathered} \tag{2.3}$$

where ε is the adiabatic cut-off † and $G(\mathbf{p},\mathbf{q})$ is continuous and bounded. Of course, in the limit $t \to +\infty$ the exponential factor goes into $\delta(E)=0$, so that for the S-operator this matrix element is zero because energy cannot be conserved in this transition. Now

$$\begin{aligned}\|\omega^{(1)}\|^2 = \int d^3p\, d^3q\, d^3p'\, d^3q'\, \delta(\mathbf{p}-\mathbf{p}')\,\delta(\mathbf{q}-\mathbf{q}')\,\delta(-\mathbf{p}-\mathbf{q}+\mathbf{p}'+\mathbf{q}') \\ \times \frac{|G(\mathbf{p},\mathbf{q})|^2}{E^2} \propto \delta(\mathbf{0}), \qquad (t<\infty)\end{aligned} \tag{2.4}$$

since $[a(\mathbf{p}),a^*(\mathbf{p}')]_\pm = \delta(\mathbf{p}-\mathbf{p}')$ for the various creation and annihilation operators. Hence $\|\omega^{(1)}\|^2$, and thus $\|\Omega\|^2$, do not exist for finite times, Q.E.D.

If we wrote instead

$$[a(\mathbf{p}),a^*(\mathbf{p}')]_\pm = \delta_T(\mathbf{p}-\mathbf{p}'), \qquad a(\mathbf{p})|0\rangle = 0 \tag{2.5}$$

where $\delta_T \neq \delta$ will be specified later, we would get

$$\begin{aligned}\langle 3|U|0\rangle = \int d^3p'\, d^3q'\, \delta_T(\mathbf{p}-\mathbf{p}')\,\delta_T(\mathbf{q}-\mathbf{q}')\,\delta_T(\mathbf{k}+\mathbf{p}'+\mathbf{q}') \times \\ \times \frac{\exp(iE't-\varepsilon|t|)}{E'}G(\mathbf{p}',\mathbf{q}')\end{aligned} \tag{2.6}$$

$$E' \equiv (\mathbf{p}'^2+m^2)^{1/2}+(\mathbf{q}'^2+m^2)^{1/2}+|\mathbf{p}'+\mathbf{q}'|$$

instead of (2.3), and

$$\begin{aligned}\|\omega^{(1)}\|^2 = \int d^3p\, d^3q\, d^3k\, d^3p_1\, d^3q_1\, d^3k_1\, \delta_T(\mathbf{p}-\mathbf{p}_1)\,\delta_T(\mathbf{q}-\mathbf{q}_1)\,\delta_T(\mathbf{k}-\mathbf{k}_1) \\ \times \overline{\langle \mathbf{p}_1,\mathbf{q}_1,\mathbf{k}_1|U^{(1)}(t,-\infty)|0\rangle}\langle \mathbf{p},\mathbf{q},\mathbf{k}|U^{(1)}(t,-\infty)|0\rangle, \qquad (t<\infty)\end{aligned} \tag{2.7}$$

† Compare, say, Schweber (1961a), p. 322.

instead of (2.4). If now $\delta_T(\mathbf{k})$ is continuous, bounded, and satisfies some integrability condition for any value of the parameter $0 < T < \infty$, the matrix element (2.6) will be a continuous density well-defined everywhere, $\|\omega^{(1)}\|^2$, (2.7), will be $< \infty$ and $\|\Omega\|^2$ will exist and equal unity for finite t if we retain the unitarity of the so-modified $U(t,t')$.

Then another desirable property follows gratis. For in (2.6) the coefficients of the exponential $\exp(iE't)$ are now continuous, bounded, etc. functions so that in the limit $t \to +\infty$, $\langle 3|U|0\rangle$ should vanish rigorously as a consequence of the Riemann-Lebesgue Lemma.† Hence we can dispense with the *ad hoc* and unsatisfactory adiabatic cut-off $\exp(-\varepsilon|t|)$. The conditions on δ_T should in fact guarantee that $\|\omega^{(1)}\|^2$, (2.7), and $\|\omega\|^2 \to 0$ as $t \to +\infty$, and that the limits $t \to +\infty$, $t' \to -\infty$ of $U(t,t')$, in particular $\lim_{t\to\infty} U(t,-\infty) = S$, exist rigorously (probably in the strong operator topology). The limit $T \to \infty$, if taken at all, must be taken after these limits $t \to +\infty$, $t' \to \infty$, perhaps only in S-matrix elements, as will be discussed more in the last section.

2.1. *Conditions on* δ_T

We shall postulate

$\delta_T(\mathbf{k})$ is a real, nonnegative, continuous, bounded function of $\mathbf{k}^2$ in $-\infty < k_i < \infty \qquad (i = 1,2,3)$ (2.1.1a)

$$\int_{-\infty}^{\infty} \delta_T(\mathbf{k})\, d^3 k = 1, \qquad \lim_{T\to\infty} \delta_T(\mathbf{k}) = \delta(\mathbf{k}) \tag{2.1.1b}$$

$\delta_T(\mathbf{k}-\mathbf{k}')$ is the kernel of a positive definite integral operator (for some function space to be specified later) (2.1.1c)

The meaning of the parameter T $(0 < T < \infty)$ will be shown to be the time-of-flight allowed for free particle momentum measurements.

For the present, these requirements seem to be necessary to achieve the cure of the U-difficulty as outlined above. We shall show below that they follow automatically from the properties of stochastic

† This states that if $f(x)$ is L^1:

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

then

$$\int_{-\infty}^{\infty} \exp(ixt) f(x)\, dx \to 0 \qquad \text{as } t \to \infty$$

space-time and the positive definite metric of state vector Hilbert Space.

3. *Stochastic Momentum Space*

We shall derive stochastic momentum space *ab initio* and later show how it follows from stochastic space-time together with $T < \infty$.

3.1. *The frequency function and its support*

The arguments parallel those for stochastic space-time (Ingraham, 1964) for the most part. The main difference is the lack of a translation group in momentum space. Thus let K^μ be the random momentum coordinate of a given free particle for frame, or observer, $\mathscr{L}$. Lorentz form-invariant theories are impossible because the frequency function is not normalizable, due to the indefinite signature of space-time. Thus one is forced mathematically to the 'three-dimensional' case: the support of K^μ is a space-like plane $\sigma(\mathscr{L})$ with normal $n(\mathscr{L})$.† This corresponds to making K^4 a certain function of the spatial momentum $\mathbf{K}$. But now—even more than in the case of position space—there are clearcut physical reasons for doing this, namely, particle energy measurements should always be reduced to the measurement of $\mathbf{K}^2$, *where we take the mass as constant by definition.* Indeed, time-of-flight measurements, which yield only the velocity $\mathbf{v}$, can never measure momentum unless we assume m is some given constant, compare Section 4. There are (at least) two reasonable assumptions:‡

$$\begin{aligned} &\mathbf{K} = \boldsymbol{\xi}, \qquad K^4 = (\mathbf{k}^2 + m^2)^{1/2} \\ &k^\mu \equiv \textit{mean}\ \text{momentum} \quad (\text{case of 'dispersionless energy'}) \end{aligned} \tag{3.1.1a}$$

$$\begin{aligned} &\mathbf{K} = \boldsymbol{\xi}, \qquad K^4 = (\boldsymbol{\xi}^2 + m^2)^{1/2} \qquad (\text{'realistic case'}) \\ &\qquad\qquad \xi \in \sigma(\mathscr{L}) \end{aligned} \tag{3.1.1b}$$

For simplicity we choose possibility (3.1.1a) in this paper.

The frequency function $f(\xi; k)$, $\xi \in \sigma(\mathscr{L})$, for the random variable $K^\mu(\xi)$ whose mean value is $k^\mu = (\mathbf{k}, \omega_\mathbf{k})$, $\omega_\mathbf{k} \equiv (\mathbf{k}^2 + m^2)^{1/2}$, with support $\sigma(\mathscr{L})$,

$$\sigma(\mathscr{L}): \qquad n(\mathscr{L}) \,.\, (\xi - k) = 0 \ \S$$

should be determined by the three principles [refer to Ingraham (1964) for a complete discussion]: (a) Relativity Principle $\equiv$ complete

† $n(\mathscr{L})$ is the unit time-like vector aligned along frame $\mathscr{L}$'s time axis.

‡ Compare the analogues for stochastic position space given in *Physical Review*, **152**, 1290, equations (1.4) and (1.5).

§ Referring the 4-vector $n(\mathscr{L})$ to its own (unprimed) frame $\mathscr{L}$: $n(\mathscr{L})^\mu = (0001)$, one sees that this is the locus $-\infty < \boldsymbol{\xi} < \infty$, $\xi^4 = k^4$ $(= \omega_\mathbf{k})$.

equivalence of all Lorentz frames $\mathscr{L}, \mathscr{L}', \mathscr{L}''$, ...; (b) Symmetry group $P \equiv$ Poincaré group; (c) K^1, K^2, K^3 are stochastically independent. For the moment we assume that $f(\xi; k) = f(\xi - k)$, although whether this form is necessary (remembering that there is no translation group in momentum space) needs further elucidation. Invariance under the subgroup of homogeneous Lorentz transformations then gives $f(\xi - k)$ is a function only of

$$(\xi - k)^2 \equiv (\boldsymbol{\xi} - \mathbf{k})^2 - (\xi^4 - k^4)^2$$

Then it will turn out that $f =$ gaussian with standard deviation a, say.

3.2. *Application to QFT*

Here we shall try to incorporate stochastic momentum space into QFT while neglecting the already known modifications† due to stochastic space-time. This is purely to shorten and simplify the exposition. It is almost certain that such a simplified treatment cannot be completely satisfactory—indeed we have very good physical reasons to believe that stochastic position space ($\lambda > 0$) is a prerequisite for stochastic momentum space, compare the Introduction. So our hope is that the properties derived here will survive *grosso modo*, unchanged in essence, in the full scale treatment.

To make the relativistic covariance of the theory—that is, to repeat, the equivalence of all Lorentz frames—and the transformation properties under the Poincaré group clearer, we shall write down in parallel the formulas for any two frames $\mathscr{L}'$ and $\mathscr{L}$ whose coordinates x'^μ and x^μ are connected by $x' = Lx \equiv \Lambda x + a$.

Consider a scalar free field Φ in ordinary QFT; referred respectively to the frames $\mathscr{L}$ and $\mathscr{L}'$, it has the momentum space expansions

$$\Phi(x) = (2\pi)^{-3/2} \int d^4 k A(k)\, \delta(k^2 + m^2)\, \theta(k) \exp(ik \,.\, x) + H\,.\,c \qquad (3.2.1)$$

$$\Phi'(x') = (2\pi)^{-3/2} \int d^4 k'\, A'(k')\, \delta(k'^2 + m^2)\, \theta(k') \exp(ik' \,.\, x') + H\,.\,c$$

$$(x' = Lx \equiv \Lambda x + a) \qquad (3.2.1')$$

where $A(k)$ and $A'(k')$ are the familiar 'four-dimensional' annihilation operators. We have‡

(1) Scalarity:

$$\Phi(x) = \Phi'(x') \Leftrightarrow A'(k) = \exp(-ik \,.\, a)\, A(k\Lambda) \qquad (3.2.2a)$$

† See footnote (†) on p. 193.

‡ Our matrix convention is $(k\Lambda)_\mu \equiv k_\nu \Lambda^\nu{}_\mu$ and $k \,.\, a = k_\mu a^\mu$. Here $A'(k)$ and $\Phi'(x)$ mean $A'(k')|_{k'=k}$ and $\Phi'(x')|_{x'=x}$ respectively.

(2) Relativistic invariance (in 'passive form'):

$$\Phi'(x) = U(L)\,\Phi(x)\,U(L)^{-1} \Leftrightarrow A'(k) = U(L)\,A(k)\,U(L)^{-1} \quad (3.2.2b)$$

(3) Combination of (1) and (2) (relativistic invariance in 'active form'):

$$\Phi(L^{-1}x) = U(L)\,\Phi(x)\,U(L)^{-1} \Leftrightarrow U(L)\,A(k)\,U(L)^{-1} = \exp(-ik\,.\,a)\,A(k\Lambda) \quad (3.2.2c)$$

where $U(L)$ is the representation of P on $\mathscr{H}$ determined by the algebra of the $A(k)$, $A^*(k)$.

Now let $k \to K$, a random variable; the $A(K)$ becomes an operator (distribution)-valued function of a random variable. The mean value $\equiv a(k;\mathscr{L})$ is computed as usual:†

$$a(k;\mathscr{L}) \equiv \int_{-\infty}^{\infty} d^3\xi f(\xi - k)\,A(K(\xi)), \qquad \xi^4 = k^4 \equiv \omega_{\mathbf{k}} \quad (3.2.3)$$

Similarly

$$a'(k';\mathscr{L}') \equiv \int_{-\infty}^{\infty} d^3\xi' f(\xi' - k')\,A'(K(\xi')), \qquad \xi'^4 = k'^4 \equiv \omega_{\mathbf{k}} \quad (3.2.3')$$

where by (3.1.1a) $\mathbf{K}(\xi) = \boldsymbol{\xi}$, $K^4(\xi) = \omega_k$ and $\mathbf{K}(\xi') = \boldsymbol{\xi}'$, $K^4(\xi') = \omega_{\mathbf{k}'}$. In these integrals, since $\xi^4 - k^4 = 0$, $\xi'^4 - k'^4 = 0$, f becomes a function of $(\boldsymbol{\xi} - \mathbf{k})^2$ and $(\boldsymbol{\xi}' - \mathbf{k}')^2$ respectively.

Incidentally, thus we must have $A(k)$ defined off the mass shell [but only slightly, for distances $k_\perp{}^2 \equiv k^2 + [k\,.\,n(\mathscr{L})]^2 \sim a^2$, compare the analogous discussion for position space in Ingraham, R. L. *Nuovo cimento*, **24**, 1117 (1962); **27**, 303 (1963); and Bailey, D. & Ingraham, R. L. (1966). *Physical Review*, **152**, 1290] if we use the simpler random variable (3.1.1a). For the more sophisticated (3.1.1b) we need only mass shell values of $A(k)$.

We can now define frame-dependent fields $\phi(x;\mathscr{L})$ and $\phi'(x';\mathscr{L}')$ in position space by Fourier transforming with respect to the *mean* momenta k and k' [$\equiv$ make the replacements $A(k) \to a(k;\mathscr{L})$ and $A'(k') \to a'(k';\mathscr{L}')$ in (3.2.1)].

3.3. *Transformation properties*

Having the $a(k;\mathscr{L})$ defined for every observer $\mathscr{L}$, and knowing the transformation properties (3.2.2a–c) of the $A(K)$, we can now derive the analogue of (3.2.2a–c) for the mean operators.

† See footnote (§) on p. 197.

Take (3.2.2b) first. We know already that since we have treated any two frames $\mathscr{L}$, $\mathscr{L}'$ exactly the same, the Relativity Principle will be satisfied. Now very general arguments,† irrespective of any dynamical details, show that the expression of the Relativity Principle for fields (here in momentum space) must be

Relativistic invariance (in 'passive form'):

$$a'(k;\mathscr{L}') = U(L)\,a(k;\mathscr{L})\,U(L)^{-1}, \qquad \mathscr{L}' = L^{-1}\mathscr{L} \tag{3.3.1}$$

where

$$\mathscr{L}' = L^{-1}\mathscr{L} \Leftrightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu} \tag{3.3.2}$$

Equation (3.3.1) is easily verified on the explicit expressions by conjugating (3.2.3) by $U(L)$, using $U(L)A(K(\xi))\,U(L)^{-1} = A'(K(\xi))$ from (3.3.2b), and noticing that the result is (3.2.3′) for the value $k' = k$.

As for (3.2.2a), its analogue is

Scalarity:

$$a'(k;\mathscr{L}') = \exp(-ik\,.\,a)\,a(k\Lambda;\mathscr{L}') \tag{3.3.3}$$

which corresponds to the scalarity of the field $\phi(\mathscr{L}')$ for any frame $\mathscr{L}'$, namely

Scalarity:

$$\phi'(x';\mathscr{L}') = \phi(x;\mathscr{L}') \tag{3.3.4}$$

For each frame $\mathscr{L}'$ is assigned a mean field: $\mathscr{L}' \to \phi(\mathscr{L}')$, and each of these mean fields will be a *scalar*, that is, have the same value at any given event referred to any frame. We define the components of $\mathscr{L}'$'s field $a(\mathscr{L}')$, referred to the (unprimed) frame $\mathscr{L}$, namely $a(k;\mathscr{L}')$, as follows:

$$a(k;\mathscr{L}') \equiv \int_{\sigma(\mathscr{L}')} d\mu(\xi-k)f(\xi-k)\exp[-i(K(\xi)-k)\,\Lambda^{-1}a]\,A(K(\xi)) \tag{3.3.5}$$

Here $\sigma(\mathscr{L}')$ is $\mathscr{L}'$'s support:

$$\sigma(\mathscr{L}'): \qquad n(\mathscr{L}')^{\mu}(\xi_{\mu}-k_{\mu}) = 0, \tag{3.3.6}$$

$n(\mathscr{L}')_{\mu}$ [$\neq$(0001) in general], k_{μ}, and ξ_{μ} are components referred to frame $\mathscr{L}$; $d\mu(\xi-k)$ is the volume element which $= d^3\xi'$ when written

† See Ingraham, R. L. (1962). *Nuovo cimento*, **26**, 328, Section 3, and Appendix 2 of *Renormalization Theory of Quantum Field Theory with a Cut-off*, to be published by Gordon and Breach.

in terms of $\mathscr{L}'$'s coordinates; and a is the unique translation vector occurring in the connection (3.3.2) of $\mathscr{L}$ and $\mathscr{L}'$'s coordinates.†

Thus $a(k;\mathscr{L}')$ is essentially the mean value of $A(K)$, the original field referred to the frame $\mathscr{L}$, averaged over $\mathscr{L}'$'s support. But in addition there is a phase factor depending on the *displacement* of the origins of the two frames, which is new compared to the case of stochastic space-time. Naturally $a(k;\mathscr{L}') \to a(k;\mathscr{L})$ given by (3.2.3) if $\mathscr{L}' = \mathscr{L}$. Note in particular that there is no phase factor in this case since $a = 0$.

From (3.3.5) one sees that only frames differing by a *spatial* rotation have the same mean field—in particular translated frames have different mean fields—in the precise sense that if $a(\mathscr{L})$ and $a(\mathscr{L}')$ are referred to some common frame, say the unprimed one, then they are identical: $a(k;\mathscr{L}) = a(k;\mathscr{L}')$ if and only if $\mathscr{L}$ and $\mathscr{L}'$ differ by a spatial rotation.

Now one can verify (3.3.3) using the definition (3.3.5).‡ Then combining (3.3.1) and (3.3.3) one gets

Relativistic invariance in 'active form':

$$U(L)\,a(k;\mathscr{L})\,U(L)^{-1} = \exp(-ik\,.\,a)\,a(k\Lambda;\mathscr{L}'), \qquad \mathscr{L}' = L^{-1}\mathscr{L} \tag{3.3.7}$$

3.4. *The commutation relations*

On the mass shell one has for the old sharp operators

$$[A(k), A^*(q)] = 2\omega_{\mathbf{k}}\,\delta(\mathbf{k}-\mathbf{q}) \tag{3.4.1}$$

† This translation a attached to observer $\mathscr{L}'$ is of a slightly more complicated nature than the 4-vector $n(\mathscr{L}')$. Both are only properties of $\mathscr{L}'$ *relative to some other frame, say* $\mathscr{L}$, because they do not assume any numerical values until one refers their components to some frame $\mathscr{L}$, obtaining $a(\mathscr{L}')^{\mu}$ and $n(\mathscr{L}')^{\mu}$. But the $a(\mathscr{L}')^{\mu}$ depend further on the relative positions of the two origins, whereas the $n(\mathscr{L}')^{\mu}$ are common to the whole class of translated frames. Thus $a(\mathscr{L}')$ should more properly be written $a(\mathscr{L}';\mathscr{L})$.

‡ Put $k\Lambda$ for k and substitute $\xi = \xi'\Lambda$ in (3.3.5) (see footnote (‡) on p. 198). Then use $f[(\xi'-k)\Lambda] = f(\xi'-k)$, $K(\xi'\Lambda) = K(\xi')\Lambda$, and $d\mu[(\xi'-k)\Lambda] = d^3\xi'$. Substituting $\xi_\mu = \xi'_\nu\Lambda^\nu{}_\mu$ into (3.3.6), this becomes the locus $-\infty < \boldsymbol{\xi}' < \infty$, $\xi'^4 = k'^4$ $(=\omega_{\mathbf{k}'})$ in virtue of $\Lambda^\nu{}_\mu n(\mathscr{L}')^\mu = n(\mathscr{L})^\nu = (0001)$ (i.e., the new integration variables ξ' are momenta referred to frame $\mathscr{L}'$). Thus multiplying by $\exp(-ik\,.\,a)$, we get

$$\exp(ik\,.\,a)\,a(k\Lambda;\mathscr{L}') = \int_{-\infty}^{\infty} d^3\xi' f(\xi'-k)\exp(-K(\xi')\,.\,a)\,A(K(\xi')\Lambda) \tag{1}$$

By scalarity of the old sharp fields [≡ first equation of equation (3.2.2)] the last two factors of the integrand $=A'(K(\xi'))$. But then (1) becomes (3.2.3′) for $k' = k$, which proves (3.3.3), Q.E.D.

Assume that they have been extended slightly off the mass shell in such a way that (3.4.1) continues to hold. Then the 'three-dimensional amplitudes'

$$a(\mathbf{k};\mathscr{L}) \equiv (2\omega_{\mathbf{k}})^{-1/2}\, a(k;\mathscr{L})$$

and correspondingly for $A(\mathbf{k})$ are connected by

$$a(\mathbf{k};\mathscr{L}) = \int_{-\infty}^{\infty} d^3\xi f(\boldsymbol{\xi}-\mathbf{k})\, A(\boldsymbol{\xi}) \tag{3.4.2}$$

from (3.2.3), where we have written $f(\boldsymbol{\xi}-\mathbf{k})$ to emphasize that on the plane $\sigma(\mathscr{L})$ it is a function only of $\boldsymbol{\xi}-\mathbf{k}$, in fact, of the 3-rotation invariant $(\boldsymbol{\xi}-\mathbf{k})^2$. Then

$$\begin{aligned}[a(\mathbf{k};\mathscr{L}), a^*(\mathbf{q};\mathscr{L})] &= \int d^3\xi\, d^3\rho f(\boldsymbol{\xi}-\mathbf{k}) f(\boldsymbol{\rho}-\mathbf{q})\, \delta(\boldsymbol{\xi}-\boldsymbol{\rho}) \\ &= \int d^3\xi f(\boldsymbol{\xi}-\mathbf{k}) f(\boldsymbol{\xi}-\mathbf{q}) \\ &= \int d^3\xi f(\boldsymbol{\xi}) f(\boldsymbol{\xi}+\mathbf{k}-\mathbf{q}) \\ &\equiv \delta_T(\mathbf{k}-\mathbf{q})\end{aligned} \tag{3.4.3}$$

From this form one can verify the properties postulated in (2.1.1a–c). Property (2.1.1a) follows from the fact that the frequency function $f(\boldsymbol{\xi})$ itself has these properties and is integrable in $-\infty < \boldsymbol{\xi} < \infty$. To verify (2.1.1b), integrate (3.4.3) with respect to $\mathbf{k}-\mathbf{q}$ and use $\int d^3\xi f(\boldsymbol{\xi}) = 1$. As for the second part of (2.1.1b), one will require of f that it gets arbitrarily narrow as $T \to \infty$, thus for $|\mathbf{k}-\mathbf{q}| > 0$ the overlap of the two factors in the last integral in (3.4.3) can be made as small as desired in this limit. Thus we shall have $\delta_T(\mathbf{k}-\mathbf{q}) \to \delta(\mathbf{k}-\mathbf{q})$, $T \to \infty$. Property (2.1.1c) follows automatically, given the connection (3.4.2), as will be elucidated in Section 5.

3.5. *Unitary of $U(t,t';\mathscr{L})$*

Remember that we formed the position space mean fields $\phi(x;\mathscr{L})$ by Fourier transforming the $a(k;\mathscr{L})$ with respect to the mean momentum k, which amounts to replacing $A(k)$ by $a(k;\mathscr{L})$ in (3.2.1). Now $U(t,t')$ is the T-exponential (2.1), where $\mathscr{H}_I(x) \equiv \mathscr{H}_I[\Phi(x)]$ is some polynomial in the fields $\Phi(x)$, ... and their derivatives. Our tentative prescription for the stochastic U operator $U(t,t';\mathscr{L})$ is to make the replacement $\Phi(x) \to \phi(x;\mathscr{L})$ in $\mathscr{H}_I[\Phi(x)]$ in the expression (2.1). At the same time T must be understood as $T_{\mathscr{L}} \equiv$ time-ordering with respect to $t \equiv \mathscr{L}$'s time coordinate.† *Then since* $\mathscr{H}_I[\phi(x;\mathscr{L})]$ *is still a self-adjoint operator,*‡ $U(t,t';\mathscr{L})$ *is unitary.*

But this definition gives exactly the U-matrix elements modified as proposed in Section 2. Namely, there also we proposed simply to replace $A(k)$ by operators (now identified as $a(k;\mathscr{L})$) having the modified commutation relations (2.5) in the usual $U(t,t')$. Hence we know that the matrix elements of our stochastic $U(t,t';\mathscr{L})$ exist, they are free from the 'U-difficulty'.

We can now see explicitly how the spatial translation invariance of $U(t,t';\mathscr{L})$ is broken. This operator is a functional of the operators $a(k;\mathscr{L})$, $a^*(k;\mathscr{L})$, $k \in$ mass shell, with complex number coefficients depending on t and t'. Then by (3.3.7) specialized to L = spatial translation a,

$$U(\mathbf{a})\, U(t,t';\mathscr{L})\, U(\mathbf{a})^{-1} = U(t,t';\mathscr{L}')$$

where (3.5.1)

$$\mathbf{x}' = \mathbf{x} + \mathbf{a}, \qquad x'^4 = x^4$$

where $U(t,t';\mathscr{L}')$ is the same functional of $a(k;\mathscr{L}')$, $a^*(k;\mathscr{L}')$, the phase factors $\exp(-i\,\mathbf{k}.\mathbf{a})$ cancelling out completely, as we illustrate below. But since $U(t,t';\mathscr{L}') \neq U(t,t';\mathscr{L})$ since $a(k;\mathscr{L}') \neq a(k;\mathscr{L})$,

$$[U(\mathbf{a}), U(t,t';\mathscr{L})] \neq 0, \qquad \text{(Q.E.D.)} \tag{3.5.2}$$

It may give insight to do this calculation explicitly. Take the vacuum-3 particle component of $U(t,-\infty;\mathscr{L})$ in order $0(e)$ as we did in Section 2, and conjugate it by $U(\mathbf{a})$. Neglecting irrelevant factors, this is

$$U(\mathbf{a}) \int d^3p\, d^3q\, d^3k\, \delta(\mathbf{p}+\mathbf{q}+\mathbf{k})\, a^*(\mathbf{p};\mathscr{L})\, b^*(\mathbf{q};\mathscr{L})\, c^*(\mathbf{k};\mathscr{L}) \times \frac{\exp(iEt)}{E} G(\mathbf{p},\mathbf{q})\, U(\mathbf{a})^{-1} \tag{3.5.3}$$

[E as in (2.3), a^*, b^*, c^* create an e^-, e^+ and a photon, respectively] where $a^*(p;\mathscr{L})$, etc., are given by (3.4.2). Now

$$U(\mathbf{a})\, a(\mathbf{p};\mathscr{L})\, U(\mathbf{a})^{-1} = \exp(-i\mathbf{p}\,.\,\mathbf{a})\, a(\mathbf{p};\mathscr{L}'), \text{ etc.}$$

where $\mathscr{L}'$'s coordinates are $\mathbf{x}' = \mathbf{x} + \mathbf{a}$, $x'^4 = x^4$. One therefore gets in the integral (3.5.3) the factor $\exp i(\mathbf{p}+\mathbf{q}+\mathbf{k}).\mathbf{a}$ which $= 1$ in virtue

† For since microcausality ('locality') is slightly violated, $T_{\mathscr{L}}$ and $T_{\mathscr{L}'}$ for different frames $\mathscr{L}$, $\mathscr{L}'$ are not in general equivalent. This definition with $T_{\mathscr{L}}$ preserves Bogoliubov causality. See Bailey, D. and Ingraham, R. (1966). *Physical Review*, **152**, 1290.

‡ Notice that it is in fact an ordinary unbounded operator whereas $\mathscr{H}_I[\Phi(x)]$ was only an operator distribution.

of the δ-function. Classically this means that $U(\mathbf{a})$ commutes with the component shown in (3.5.3); but now we have that the resulting creation operators are in fact $a^*(\mathbf{p};\mathscr{L}')$, etc., rather than $a^*(\mathbf{p};\mathscr{L})$, etc.

4. *Derivation of $f(\xi - k)$ from Time-of-Flight Measurements*

In this section we show that stochastic position space together with a finite time T for momentum measurements by the time-of-flight method imply a stochastic momentum space, and derive its particular form. The resulting realization is in accord with the abstract theory developed independently in the last section.

For simplicity we confine ourself here to nonrelativistic velocities. The relativistic case has been done and will be given elsewhere.

By the time-of-flight method of measuring the momentum of a *free* particle, one measures the particle's positions $\mathbf{x}_1$ and $\mathbf{x}_2$ at two times, t_1 and t_2 respectively. This determines the velocity $\mathbf{v}$, and, *given the mass,* the spatial momentum $\mathbf{k}$. Thus when space-time becomes random: $x \to X \equiv$ random space-time variable, the 3-momentum $\mathbf{K}$, defined by

$$\mathbf{K} \equiv m(\mathbf{X}_1 - \mathbf{X}_2)/(X_1{}^4 - X_2{}^4) \tag{4.1}$$

becomes random, a function of two independent random variables $X_1{}^\mu$, $X_2{}^\mu$. From now on we fix the Lorentz frame, call it $\mathscr{L}$, the unprimed frame. The frequency function f of a random variable defined as a function of one or more random variables is obtained by standard rules:†

$$\begin{aligned} f(\boldsymbol{\xi};\mathbf{k}) = \int_{-\infty}^{\infty} & d^3X_1\,d^3X_2\,p(\mathbf{X}_1 - \mathbf{x}_1)\,p(\mathbf{X}_2 - \mathbf{x}_2) \\ & \times \delta[\boldsymbol{\xi} - m(\mathbf{X}_1 - \mathbf{X}_2)/(X_1{}^4 - X_2{}^4)] \end{aligned} \tag{4.2}$$

Here $f(\boldsymbol{\xi};\mathbf{k}) \equiv f(\xi;k)$, the momentum frequency function, defined on the plane $\sigma(\mathscr{L})$: $-\infty < \boldsymbol{\xi} < \infty$, $\xi^4 = \omega_{\mathbf{k}}$. $p(\mathbf{X} - \mathbf{x}) \equiv g(X - x)$† is the frequency function of the random space-time coordinate X^μ of mean value x^μ defined on the plane $-\infty < \mathbf{X} < \infty$, $X^4 = x^4$. (We are adopting here the case of 'dispersionless time' for simplicity, although the more realistic case of the 'Einstein clock',‡ namely $X^4 = |\mathbf{X} - \mathbf{x}| + x^4 + \text{const.}$, would undoubtedly be better.)

† See any book on random variable theory. We change notation slightly from that of Ingraham (1964).

‡ Compare *Physical Review*, **152**, 1290, equations (1.4) and (1.5).

First, we can verify $\int d^3\xi f(\boldsymbol{\xi};\mathbf{k}) = 1$ directly on (4.2).

Now, our basic assumption is that we use the fixed finite time-of-flight T for all momentum measurements. This means that the *measured* values ($\equiv$mean values) $x_1{}^4$, $x_2{}^4$ of the two times satisfy

$$x_1{}^4 - x_2{}^4 = T \tag{4.3}$$

Since we are using the simplified case of dispersionless time, we can directly substitute T for $X_1{}^4 - X_2{}^4$ in the δ-function in (4.2).

Then we proceed to manipulate (4.2), using at the end the known form of $p(\mathbf{X}-\mathbf{x})$. Make the change of variables $\mathbf{X}_1' = \mathbf{X}_1 - \mathbf{X}_2$, $\mathbf{X}_2' = \mathbf{X}_2 - \mathbf{x}_2$, integrate out the δ-function, and drop primes:

$$f(\boldsymbol{\xi};\mathbf{k}) = \left(\frac{T}{m}\right)^3 \int d^3X_2\, p\left(\mathbf{X}_2 + \mathbf{x}_2 - \mathbf{x}_1 + \frac{T}{m}\boldsymbol{\xi}\right) p(\mathbf{X}_2) \tag{4.4}$$

From the theory of stochastic space-time

$$p(\mathbf{X}-\mathbf{x}) = N\exp[-(\mathbf{X}-\mathbf{x})^2/2\lambda^2], \qquad N \equiv [\surd(2\pi)\lambda]^{-3} \tag{4.5}$$

We shall need the composition law of gaussians:

$$N^2 \int d^3X \exp[-(\mathbf{X}-\mathbf{x})^2/2\lambda^2] \exp(-\mathbf{X}^2/2\lambda^2) = 2^{-3/2} N \exp(-\mathbf{x}^2/4\lambda^2) \tag{4.6}$$

Thus

$$\begin{aligned} f(\boldsymbol{\xi};\mathbf{k}) &= f(\boldsymbol{\xi}-\mathbf{k}) = [\surd(2\pi)a]^{-3}\exp[-(\boldsymbol{\xi}-\mathbf{k})^2/2a^2] \\ a &\equiv \surd 2\, m\lambda/T, \qquad \mathbf{k} \equiv m(\mathbf{x}_1 - \mathbf{x}_2)/T \end{aligned} \tag{4.7}$$

So f itself is a three-dimensional gaussian with the expected mean value $\mathbf{k}$. We see that the standard deviation $a \propto \lambda/T$, as expected from the physical arguments adumbrated in the Introduction. Further, we see that $a \propto m$, which merely reflects the obvious fact that for a given velocity uncertainty, the uncertainty in momentum is directly proportional to the mass. Finally, note that $f(\boldsymbol{\xi};\mathbf{k})$ is indeed a function of the difference $\boldsymbol{\xi}-\mathbf{k}$.

4.1. *Explicit form of* δ_T

We can calculate the specific form of δ_T using the general formula (3.4.3) and the specific form of f found here. The gaussian composition law (4.6) yields

$$\begin{aligned} \delta_T(\mathbf{k}-\mathbf{q}) &= [\surd(2\pi)b]^{-3}\exp[-(\mathbf{k}-\mathbf{q})^2/2b^2] \\ b &\equiv 2m\lambda/T \end{aligned} \tag{4.1.1}$$

Now $f(\boldsymbol{\xi}-\mathbf{k})$, (4.7), obeys all the postulates of stochastic momentum space developed independently in Section 3 (compare the remarks following equation (3.1.1.)], hence, as proved in Section 3 [equation (3.4.3) ff.], δ_T given by (4.1.1) has the properties (2.1.1). In particular note that $\delta_T \to \delta$ as $T \to \infty$ as the gaussian approximation to the δ-function. Further, although we *know* that the spectrum of this $\delta_T(\mathbf{k}-\mathbf{q})$ is positive (compare Section 5), we derive it anyway for later use.

Recall that the Fourier transform of a gaussian is a gaussian. We can write this in the following way:

$$\int d^3 q\, \delta_T(\mathbf{k}-\mathbf{q}) \exp(i\mathbf{q}\,.\,\mathbf{x}) = \exp(-\mathbf{x}^2 b^2/2) \exp(i\mathbf{k}\,.\,\mathbf{x}) \qquad (4.1.2)$$

Thus the corresponding integral operator has 'non-normalizable eigenvectors' $\exp(i\mathbf{k}\,.\,\mathbf{x})$, $-\infty < \mathbf{k} < \infty$, labelled by any 3-vector $\mathbf{x}$ and corresponding 'eigenvalues' $\exp(-\mathbf{x}^2 b^2/2) > 0$. Thus it is positive definite.

5. '*Relatively Free*' *Particle States*

If we build stochastic QFT as in Section 3, the state vector Hilbert Space is unchanged: it is still that space generated from the vacuum by the old sharp operator algebra of the $A(k)$, $A^*(k)$. However, let us look at the properties of the 'relatively free' particle states $|\mathbf{k}_1, \mathbf{k}_2, \ldots \mathbf{k}_n; \mathscr{L}\rangle$ generated by applying the mean creation operators $a^*(\mathbf{k};\mathscr{L})$ a number of times to the vacuum. These are *bona-fide* ($\equiv$normalizable) states. There is an ensemble of these states for each $\mathscr{L}$ and they are generally different for different $\mathscr{L}$. Now since we still have $a(\mathbf{k};\mathscr{L})|0\rangle = 0$, any $\mathbf{k}$, any $\mathscr{L}$, n-particle states are still strictly orthogonal to m-particle states for $n \neq m$. The new feature is that n-particle states with different sets of mean momenta are now not strictly orthogonal:

$$\begin{aligned} \langle \mathscr{L}; \mathbf{k} | \mathbf{q}; \mathscr{L}\rangle &= \langle 0 | [a(\mathbf{k};\mathscr{L}), a^*(\mathbf{q};\mathscr{L})] | 0\rangle \\ &= \delta_T(\mathbf{k}-\mathbf{q}) \end{aligned} \qquad (5.1)$$

Incidentally, if $\psi \equiv \int d^3 k c(\mathbf{k}) |\mathbf{k};\mathscr{L}\rangle$ for some complex coefficients $c(\mathbf{k})$, the Hilbert Space requirement $\|\psi\|^2 \geqslant 0$ gives

$$\int d^3 k\, d^3 q \overline{c(\mathbf{k})}\, c(\mathbf{q})\, \delta_T(\mathbf{k}-\mathbf{q}) \geqslant 0 \qquad (5.2)$$

for any $c(\mathbf{k})$, hence the requirement that the kernel $\delta_T(\mathbf{k}-\mathbf{q})$ be positive definite. This is the origin of the requirement (2.1.1c), where the Hilbert Space (i.e., the inner product) was simply determined by the two properties (2.5). However, note that if we derive the $a(k;\mathscr{L})$ as the averages of the $A(\xi)$ as in Section 3, $\mathscr{H}$ is already determined by the commutation relations of the $A(\xi)$ and $A^*(\xi)$ and $A(\xi)|0\rangle = 0$ and we know it is a Hilbert Space. Thus we know $\|\psi\|^2 \geqslant 0$ and hence from (5.2) we know that δ_T given by (3.4.3) is positive definite.

About these ensembles $E(\mathscr{L})$ of relatively free n-particle states we shall need to require that they are at least linearly independent and form a basis for $\mathscr{H}$ in the sense that $\mathscr{H}$ = closure of the finite complex linear span of $E(\mathscr{L})$ for each $\mathscr{L}$. From Lorentz invariance in the 'active form' (3.3.7) these bases are transformed into each other (up to phase factors) by the action of the Poincaré group, e.g.

$$U(L)|k;\mathscr{L}\rangle = \exp(ik\,.\,a)|k\Lambda;\mathscr{L}'\rangle, \qquad \mathscr{L}' = L^{-1}\mathscr{L} \tag{5.3}$$

Finally, a most important point: $|\mathbf{k}_1, \mathbf{k}_2, \ldots \mathbf{k}_n; \mathscr{L}\rangle$ *will represent that physical n-particle state for which observer* $\mathscr{L}$ *measures the mean momenta* $\mathbf{k}_1, \mathbf{k}_2, \ldots \mathbf{k}_n$. These states are then the physical ones, between which we want to compute transition probabilities, *not* the old orthogonal states $A^*(\mathbf{k}_1) \ldots A^*(\mathbf{k}_n)|0\rangle$, which become mathematical constructs with no direct physical meaning. Indeed, in Section 2 it was the matrix element of $U(t,t';\mathscr{L})$ between these relatively free particle states that we computed.

The name is justified because if momentum space is random, we cannot distinguish a really free particle from one acted on by an interaction sufficiently weak to give a change in momentum of the same order as the uncertainty a.

6. *Inequalities and Limits*

As one sees from the foregoing, $U(t,t';\mathscr{L})$ is well defined for any $T < \infty$. Thus the formalism will give an answer for the transition probability between an initial state of n_i particles of mean momenta $\{p_1, p_2, \ldots p_{n_i}\} \equiv P_i$ 'at time t'' to a final state of n_f particles of mean momenta P_f† 'at the later time t' namely

$$P_{f\leftarrow i} \equiv |\langle\mathscr{L};P_f|U(t,t';\mathscr{L})|P_i;\mathscr{L}\rangle|^2 \tag{6.1}$$

whatever T.

However, this well defined number may bear little relation to experiment unless T is chosen in an optimum way [for each experiment, i.e., each particular U-matrix element (6.1)], owing to the basic fact of random variable theory that the larger the dispersion, the poorer is the mean value as an approximation to the result of any individual

† In ordinary QFT ($\lambda = 0$) these constant eigenstates of the total 4-momentum P_μ, etc. represent only the instantaneous momentum of the particles at the finite times t and t', of course.

measurement. In this final section we therefore give some preliminary thinking on this question; we do not expect to get to the bottom here of what is undoubtedly a deep subject.

In the current QFT ($\lambda = 0$) there is no trouble defining the state of momentum p 'at time t'. This is in fact just the basic idea of the differential calculus. One must take T sufficiently short that the change in p due to the interaction $\equiv (\Delta p)_{\text{int}}$ is much less than the mean p determined by the time-of-flight method [defined in equation (4.7)]:

$$(\Delta p)_{\text{int}} \ll p_{\text{mean}} \tag{6.2}$$

Since space-time can be measured arbitrarily sharply, the shorter T is, the better. This is expressed by $T_{\text{opt}} = 0$.

However, in the stochastic QFT presented here ($\lambda > 0$) this situation is changed. Just as in current QFT one has the inequality (6.2), establishing an upper bound on T (depending on the details of the interaction, the experiment in question, and so on, which we shall not try to estimate here). But T must not be taken too short either, or else the error $(\Delta p)_{\text{stoch}}$ in p due to the stochastic momentum space: $(\Delta p)_{\text{stoch}} \approx a$, the standard deviation, will be of the same order as p_{mean}. This yields the inequality $a \ll p_{\text{mean}}$, or by (4.7)

$$\lambda \ll \Delta x \tag{6.3}$$

where $\Delta x \equiv |\mathbf{x}_1 - \mathbf{x}_2|$. Thus, irrespective of the details of the experiment, the interaction, etc., the 'distance of flight' Δx must be much greater than the intrinsic space-time dispersion λ. This yields a lower bound on T depending on the velocities, etc. Hence there is some optimum time-of-flight $T_{\text{opt}} > 0$ for each U-matrix element.

Let us raise another question: what is the relation between the time magnitudes T and $t - t'$? Briefly, it seems that there is no relation. The reasoning is as follows.

There are three *disjoint* time intervals: first one prepares the initial state $\equiv$ measures the initial mean momenta P_i in an interval I_i of length T; next, the particles are allowed to interact during the interval $I_{\text{int}} \equiv (t,t')$; finally the final mean momenta P_f are measured in an interval I_f of length T. These intervals must not overlap or else by the basic postulates of quantum mechanics the matrix element (6.1) will not represent the probability $P_{f\leftarrow i}$; i.e., $U(t,t';\mathscr{L})$ represents the time evolution of the system between t' and t only if no measurements are made within that interval. Now just because they do not overlap, we can see no physical reason for any necessary, general inequalities between them. It seems, on the contrary, that t and t' should be freely assignable as *part* of the data defining the desired experiment; and once this experiment is fixed, T is determined independently as T_{opt} by other data: the size of λ, the strength of the interaction, the velocities, etc., as argued above.

The only limitation of this sort is given by the mathematics: we know that for finite t or t' one cannot take the limit $T \to \infty$ or else one falls back into the U-difficulty (e.g., lim $\|\omega^{(1)}\|^2, T \to \infty$, does not exist, compare equation (2.7)). The physical interpretation of this fact seems to be that we do not have an arbitrarily long time to measure initial or final momenta if the experiment begins or ends at a finite time. This in spite of the fact that we saw no physical reason for a dependence between T and $t - t'$ just above.

On the other hand, as illustrated in Section 2, the limit $t \to +\infty$, $t' \to -\infty$, T finite and fixed, makes mathematical sense: this makes $U(t,t';\mathscr{L}) \to S(\mathscr{L})$, a well-defined limit. Whether the subsequent limit

$$\tilde{S} \equiv \lim_{T \to \infty} S(\mathscr{L})$$

exists we do not know, but conjecture that it does, and that $\tilde{S}$ coincides with the current S except where the latter breaks down because of the U-difficulty, namely in the (practically unimportant) vacuum–vacuum matrix elements. In any case T should probably be kept finite even in the S-operator $S(\mathscr{L})$, its magnitude determined by the actual experimental conditions.

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